

2020 考研数学三真题及解析 (完整版)

来源: 文都教育

一、选择题: 1~8 小题, 第小题 4 分, 共 32 分. 下列每题给出的四个选项中, 只有一个选项是符合题目要求的, 请将选项前的字母填在答题纸指定位置上.

1. 设 $\lim_{x \rightarrow \infty} \frac{f(x)-a}{x-a} = b$, 则 $\lim_{x \rightarrow \infty} \frac{\sin f(x) - \sin a}{x-a}$

- A. $b \sin a$
- B. $b \cos a$
- C. $b \sin f(a)$
- D. $b \cos f(a)$

答案: B

解析:

$$\lim_{x \rightarrow a} \frac{\sin f(x) - \sin a}{x-a} = \lim_{x \rightarrow a} \frac{[f(x)-a]}{x-a} \cos \xi = b \cos a$$

(其中 ξ 介于 $f(x)$ 与 a 之间)

∴ 选 B

2. $f(x) = \frac{e^{\frac{1}{x-1}} \ln |1+x|}{(e^x-1)(x-2)}$ 第二类间断点个数

- A. 1
- B. 2
- C. 3
- D. 4

答案: C

解析:

$x=0, x=2, x=1, x=-1$ 为间断点

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{e^{\frac{1}{x-1}} \ln |1+x|}{(e^x-1)(x-2)} = \lim_{x \rightarrow 0} \frac{e^{-1} \ln |1+x|}{-2x} = -\frac{e^{-1}}{2} \lim_{x \rightarrow 0} \frac{\ln |x+1|}{x} = -\frac{e^{-1}}{2}$$

$x=0$ 为可去间断点

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{e^{\frac{1}{x-1}} \ln |1+x|}{(e^x-1)(x-2)} = \infty$$

$x=2$ 为第二类间断点

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \frac{e^{\frac{1}{x-1}} \ln |1+x|}{(e^x-1)(x-2)} = 0$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{e^{\frac{1}{x-1}} \ln |1+x|}{(e^x - 1)(x-2)} = \infty$$

$x = 1$ 为第二类间断点

$$\lim_{x \rightarrow -1} f(x) = \lim_{x \rightarrow -1} \frac{e^{\frac{1}{x-1}} \ln |1+x|}{(e^x - 1)(x-2)} = \infty$$

$x = -1$ 为第二类间断点

3. 设奇函数 $f(x)$ 在 $(-\infty, +\infty)$ 上具有连续导数, 则

A. $\int_0^x [\cos f(t) + f'(t)] dt$ 是奇函数

B. $\int_0^x [\cos f(t) + f'(t)] dt$ 是偶函数

C. $\int_0^x [\cos f'(t) + f(t)] dt$ 是奇函数

D. $\int_0^x [\cos f'(t) + f(t)] dt$ 是偶函数

答案: A

解析:

$$F(x) = \int_0^x [\cos f(t) + f'(t)] dt$$

$$F'(x) = \cos f(x) + f'(x)$$

由 $f(x)$ 为奇函数知, $f'(x)$ 为偶函数.

$\cos f(x)$ 为偶函数. 故 $F'(x)$ 为偶函数.

$F(x)$ 为奇数. ∴ 选 A

4. 设幂级数 $\sum_{n=1}^{\infty} na_n(x-2)^n$ 的收敛区间为 $(-2, 6)$, 则 $\sum_{n=1}^{\infty} a_n(x+1)^{2n}$ 的收敛区间为

A. $(-2, 6)$

B. $(-3, 1)$

C. $(-5, 3)$

D. $(-17, 15)$

答案: B

解析:

$$\text{由于 } \lim_{n \rightarrow \infty} \frac{(n+1)a_{n+1}}{na_n} = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \rho_1 = \frac{1}{R_1} = \frac{1}{4}$$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \rho_2 = \rho_1 = \frac{1}{4} \quad \therefore R_2 = 4.$$

$\therefore R'_2 = \sqrt{R_2} = 2$ ，故所求收敛域为 $(-3, 1)$ ，

\therefore 选 B.

5. 设 4 阶矩阵 $A = (a_{ij})$ 不可逆， a_{12} 的代数余子式 $A_{12} \neq 0$ ， $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ 为矩阵 A 的列向量

组， A^* 为 A 的伴随矩阵，则 $A^*x = 0$ 的通解为

A. $x = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3$

B. $x = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_4$

C. $x = k_1\alpha_1 + k_2\alpha_3 + k_3\alpha_4$

D. $x = k_1\alpha_2 + k_2\alpha_3 + k_3\alpha_4$

答案：C

解析：

$\because A$ 不可逆

$\therefore |A| = 0$

$\because A_{12} \neq 0 \quad \therefore r(A) = 3$

$\therefore r(A^*) = 1$

$\therefore A^*x = 0$ 的基础解系有 3 个线性无关的解向量.

$\because A^*A = |A|E = 0$

$\therefore A$ 的每一列都是 $A^*x = 0$ 的解

又 $\because A_{12} \neq 0 \quad \therefore \alpha_1, \alpha_3, \alpha_4$ 线性无关

$\therefore A^*x = 0$ 的通解为 $x = k_1\alpha_1 + k_2\alpha_3 + k_3\alpha_4$ ，故选 C.

6. 设 A 为 3 阶矩阵， α_1, α_2 为 A 的属于特征值 1 的线性无关的特征向量， α_3 为 A 的属于 -1

的特征向量, 则 $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 的可逆矩阵 P 为

A. $(\alpha_1 + \alpha_3, \alpha_2, -\alpha_3)$

B. $(\alpha_1 + \alpha_2, \alpha_2, -\alpha_3)$

C. $(\alpha_1 + \alpha_3, -\alpha_3, \alpha_2)$

D. $(\alpha_1 + \alpha_2, -\alpha_3, \alpha_2)$

答案: D

解析:

$$A\alpha_1 = \alpha_1, A\alpha_2 = \alpha_2$$

$$A\alpha_3 = -\alpha_3$$

$$\therefore P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\therefore P$ 的 1, 3 两列为 1 的线性无关的特征向量 $\alpha_1 + \alpha_2, \alpha_2$

P 的第 2 列为 A 的属于 -1 的特征向量 $-\alpha_3$.

$$\therefore P = (\alpha_1 + \alpha_2, -\alpha_3, \alpha_2)$$

\therefore 选 D

7. 设 A, B, C 为三个随机事件, 且 $P(A) = P(B) = P(C) = \frac{1}{4}$, $P(AB) = 0$, $P(AC) =$

$P(BC) = \frac{1}{12}$, 则 A, B, C 中恰有一个事件发生的概率为

A. $\frac{3}{4}$ B. $\frac{2}{3}$ C. $\frac{1}{2}$ D. $\frac{5}{12}$

答案: D

解析:

$$P(\overline{A}\overline{B}\overline{C}) = P(\overline{A}\overline{B}C) = P(A) - P[A(BUC)]$$

$$\begin{aligned}
 &= P(A) - P(AB + AC) \\
 &= P(A) + P(AB) - P(AC) + P(ABC) \\
 &= \frac{1}{4} - 0 - \frac{1}{12} + 0 = \frac{1}{6}
 \end{aligned}$$

$$\begin{aligned}
 P(B\bar{A}\bar{C}) &= P(\overline{BAUC}) = P(B) - P[B(AUC)] \\
 &= P(B) - P(BA) - P(BC) + P(ABC) \\
 &= \frac{1}{4} - 0 - \frac{1}{12} + 0 = \frac{1}{6}
 \end{aligned}$$

$$\begin{aligned}
 P(\bar{C}\bar{B}\bar{A}) &= P(\overline{CBUA}) = P(C) - P[CU(BUA)] \\
 &= P(C) - P(CB) - P(CA) + P(ABC) \\
 &= \frac{1}{4} - \frac{1}{12} - \frac{1}{12} + 0 = \frac{1}{12}
 \end{aligned}$$

$$\begin{aligned}
 P(\bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}C + \bar{A}B\bar{C}) &= P(\bar{A}\bar{B}\bar{C}) + P(\bar{A}\bar{B}C) + P(\bar{A}B\bar{C}) \\
 &= \frac{1}{6} + \frac{1}{6} + \frac{1}{12} = \frac{5}{12}
 \end{aligned}$$

8. 设随机变量 (X, Y) 服从二维正态分布 $N\left(0, 0; 1, 4; -\frac{1}{2}\right)$, 随机变量中服从标准正态分布且

与 X 独立的是

- A. $\frac{\sqrt{5}}{5}(X+Y)$ B. $\frac{\sqrt{5}}{5}(X-Y)$
 C. $\frac{\sqrt{3}}{3}(X+Y)$ D. $\frac{\sqrt{3}}{3}(X-Y)$

答案: C

解析:

$$D\left[\frac{\sqrt{3}}{3}(X+Y)\right] = \frac{1}{3}[DX + DY] + \frac{2}{3}\text{cov}(X, Y)$$

$$\begin{aligned}
 &= \frac{1}{3}[DX + DY] + \frac{2}{3}\rho\sqrt{DX} \cdot \sqrt{DY} \\
 &= \frac{5}{3} - \frac{2}{3} = 1 \\
 E\left[\frac{\sqrt{3}}{3}(X+Y)\right] &= 0 \\
 \therefore \frac{\sqrt{3}}{3}(X+Y) &\sim N(0,1).
 \end{aligned}$$

二、填空题：9-14 小题，每小题 4 分，共 24 分。请将答案写在答题纸指定的位置上

9. 设 $z = \arctan[xy + \sin(x+y)]$, 则 $dz|_{(0,\pi)} =$ _____.

解析:

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$\frac{\partial z}{\partial x} = \frac{1}{1+[xy + \sin(x+y)]^2} [y + \cos(x+y)], \quad \frac{\partial z}{\partial x}\bigg|_{(0,\pi)} = \pi - 1$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+[xy + \sin(x+y)]^2} [x + \cos(x+y)], \quad \frac{\partial z}{\partial y}\bigg|_{(0,\pi)} = -1$$

$$\therefore \frac{\partial z}{\partial x}\bigg|_{(0,\pi)} = (\pi - 1)dx - dy$$

10. 曲线 $x + y + e^{2xy} = 0$ 在点 $(0, -1)$ 处的切线方程为 _____.

解析:

$$1 + y' + e^{2xy}(2y + 2xy') = 0 \quad \text{①}$$

将 $x=0, y=-1$ 代入①得 $y' = 1 = k$.

$$\therefore y + 1 = 1(x - 0)$$

即 $y = x - 1$.

11. Q 表示产量，成本 $C(Q) = 100 + 13Q$ ，单价 P ，需求量 $Q(P) = \frac{800}{P+3} - 2$. 则工厂取得利

润最大时的产量为 _____.

解析:

$$\begin{aligned} L &= QP - C(Q) \\ &= Q\left(\frac{800}{Q+2} - 3\right) - 100 - 13Q \\ &= \frac{800Q}{Q+2} - 16Q - 100 \end{aligned}$$

$$L'(Q) = \frac{1600 - 16(Q+2)^2}{(Q+2)^2} = 0$$

$$\therefore Q = 8$$

12. 设平面区域 $D = \left\{ (x, y) \mid \frac{x}{2} \leq y \leq \frac{1}{1+x^2}, 0 \leq x \leq 1 \right\}$, 则 D 绕 y 轴旋转所成旋转体体积为

解析:

$$\begin{aligned} &\pi \int_0^{\frac{1}{2}} x^2 dy + \pi \int_{\frac{1}{2}}^1 x^2 dy \\ &= \pi \int_0^{\frac{1}{2}} 4y^2 dy + \pi \int_{\frac{1}{2}}^1 \left(\frac{1}{y} - 1\right) dy \\ &= \frac{4}{3} \pi y^3 \Big|_0^{\frac{1}{2}} + \pi \left[\ln y \Big|_{\frac{1}{2}}^1 - \frac{1}{2} \right] \\ &= \frac{4}{3} \pi \cdot \frac{1}{8} + \pi \left(\ln 2 - \frac{1}{2} \right) \\ &= \pi \left(\ln 2 - \frac{1}{3} \right) \end{aligned}$$

13. 行列式 $\begin{vmatrix} a & 0 & -1 & 1 \\ 0 & a & 1 & -1 \\ -1 & 1 & a & 0 \\ 1 & -1 & 0 & a \end{vmatrix} =$ _____.

解析:

$$\begin{aligned} & \begin{vmatrix} a & 0 & -1 & 1 \\ 0 & a & 1 & -1 \\ -1 & 1 & a & 0 \\ 1 & -1 & 0 & a \end{vmatrix} = \begin{vmatrix} a & 0 & -1 & 1 \\ 0 & a & 1 & -1 \\ -1 & 1 & a & 0 \\ 0 & 0 & a & a \end{vmatrix} \\ & = \begin{vmatrix} 0 & a & -1+a^2 & 1 \\ 0 & a & 1 & -1 \\ -1 & 1 & a & 0 \\ 0 & 0 & a & a \end{vmatrix} = - \begin{vmatrix} a & -1+a^2 & 1 \\ a & 1 & -1 \\ 0 & a & a \end{vmatrix} \\ & = - \begin{vmatrix} a & a^2-2 & 1 \\ a & 2 & -1 \\ 0 & 0 & a \end{vmatrix} = a^4 - 4a^2. \end{aligned}$$

14. 随机变量 X 的概率分布 $P\{X=k\} = \frac{1}{2^k}, k=1,2,3,\dots, Y$ 表示 X 被 3 除的余数, 则

$E(Y) =$ _____.

解析:

$$P\{Y=0\} = P\{X=3k, k=1,2,\dots\}$$

$$P\{Y=1\} = P\{X=3k+1, k=0,1,2,\dots\} = \sum_{k=0}^{\infty} \frac{1}{2^{3k+1}}$$

$$P\{Y=2\} = P\{X=3k+2, k=0,1,2,\dots\} = \sum_{k=0}^{\infty} \frac{1}{2^{3k+2}}$$

$$E(Y) = 1 \cdot \sum_{k=0}^{\infty} \frac{1}{2^{3k+1}} + 2 \cdot \sum_{k=0}^{\infty} \frac{1}{2^{3k+2}}$$

$$= \frac{1}{2} \frac{1}{1-\frac{1}{8}} + \frac{1}{2} \frac{1}{1-\frac{1}{8}}$$

$$= \frac{8}{7}$$

三、解答题: 15~23 小题, 共 94 分. 请将解答写在答题纸指定位置上. 解答写出文字说明、证明过程或演算步骤.

15. 已知 a, b 为常数, $\left(1 + \frac{1}{n}\right)^n - e$ 与 $\frac{b}{n^a}$, 当 $n \rightarrow \infty$ 时为等价无穷小, 求 a, b .

15. 【解】

$$\begin{aligned}
 1 &= \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n}\right)^n - e}{\frac{b}{n^a}} = \frac{1}{b} \lim_{n \rightarrow \infty} n^a [e^{n \ln\left(1 + \frac{1}{n}\right)} - e] \\
 &= \frac{1}{b} \cdot \lim_{n \rightarrow \infty} n^a \cdot e [e^{n \ln\left(1 + \frac{1}{n}\right) - 1} - 1] \\
 &= \frac{1}{b} \cdot \lim_{n \rightarrow \infty} n^a \cdot e \left[n \ln\left(1 + \frac{1}{n}\right) - 1 \right] \\
 &= \frac{1}{b} \lim_{n \rightarrow \infty} n^a \cdot e \left[n \left(\frac{1}{n} - \frac{1}{2n^2} \right) - 1 \right] \\
 &= \frac{1}{b} \lim_{n \rightarrow \infty} n^{a-1} \left(-\frac{1}{2} \right) e \\
 \therefore a-1 &= 0 \\
 \therefore a &= 1 \quad \frac{1}{b} \cdot \left(-\frac{e}{2} \right) = 1 \\
 b &= -\frac{e}{2}
 \end{aligned}$$

16. 求二元函数 $f(x, y) = x^3 + 8y^3 - xy$ 的极值

解析:

求一阶导可得

$$\begin{aligned}
 \frac{\partial f}{\partial x} &= 3x^2 - y \\
 \frac{\partial f}{\partial y} &= 24y^2 - x
 \end{aligned}$$

$$\text{令} \begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \text{ 可得} \begin{cases} x = 0 \\ y = 0 \end{cases} \begin{cases} x = \frac{1}{6} \\ y = \frac{1}{12} \end{cases}$$

求二阶导可得

$$\frac{\partial^2 f}{\partial x^2} = 6x \quad \frac{\partial^2 f}{\partial x^2 y} = -1 \quad \frac{\partial^2 f}{\partial y^2} = 48y$$

当 $x = 0, y = 0$ 时, $A = 0, B = -1, C = 0$

$AC - B^2 < 0$ 故不是极值.

当 $x = \frac{1}{6}, y = \frac{1}{12}$ 时

$$A = 1, B = -1, C = 4.$$

$AC - B^2 > 0, A = 1 > 0$ 故 $\left(\frac{1}{6}, \frac{1}{12}\right)$ 且极小值

$$\text{极小值 } f\left(\frac{1}{6}, \frac{1}{12}\right) = \left(\frac{1}{6}\right)^3 + 8\left(\frac{1}{12}\right)^3 - 6 \times \frac{1}{12} = -\frac{1}{216}$$

17. 若 $y'' + 2y' + 5y = 0, f(0) = 1, f'(0) = -1$, 则

(1) 求 $f(x)$

(2) $a_n = \int_{n\pi}^{+\infty} f(x) dx$, 求 $\sum_{i=1}^n a_n$

解析:

(1) $y'' + 2y' + 5y = 0$ 的特征方程为 $r^2 + 2r + 5 = 0$

$$\therefore r_{1,2} = -1 \pm 2i$$

$$\therefore y(x) = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$$

$$y'(x) = -e^{-x}(c_1 \cos 2x + c_2 \sin 2x) + e^{-x}(-2c_1 \sin 2x + 2c_2 \cos 2x)$$

$$\therefore y(0) = 1, y'(0) = -1$$

$$\therefore c_1 = 1, c_2 = 0$$

$$\therefore y(x) = e^{-x} \cos 2x$$

$$(2) a_n = \int_{n\pi}^{+\infty} f(x) dx = \int_{n\pi}^{+\infty} e^{-x} \cos 2x dx$$

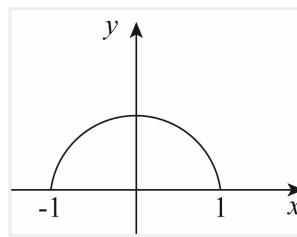
$$\begin{aligned}
 &= -\int_{n\pi}^{+\infty} \cos 2x \, d e^{-x} = -\cos 2x \cdot e^{-x} \Big|_{n\pi}^{+\infty} + \int_{n\pi}^{+\infty} e^{-x} \, d \cos 2x \\
 &= -e^{-n\pi} - 2 \int_{n\pi}^{+\infty} e^{-x} \sin 2x \, dx \\
 &= -e^{-n\pi} + 2 \int_{n\pi}^{+\infty} \sin 2x \, d e^{-x} \\
 &= -e^{-n\pi} + 2 \sin 2x e^{-x} \Big|_{n\pi}^{+\infty} - 2 + \int_{n\pi}^{+\infty} e^{-x} \cos 2x \, dx \\
 &\therefore 5a_n = -e^{-n\pi} \\
 &\therefore a_n = -\frac{1}{5} e^{-n\pi}
 \end{aligned}$$

$$\begin{aligned}
 \sum_{i=1}^n a_i &= -\frac{1}{5} [e^{-\pi} + e^{-2\pi} + \dots + e^{-n\pi}] \\
 &= -\frac{1}{5} \cdot \frac{e^{-\pi} [1 - e^{-n\pi}]}{1 - e^{-\pi}} \\
 &= -\frac{1}{5} \cdot \frac{1 - e^{-n\pi}}{e^{\pi} - 1}
 \end{aligned}$$

18. $f(x, y) = y\sqrt{1-x^2} + x \iint_D f(x, y) \, dx \, dy$ 其中

$$D = \left\{ (x, y) \mid \begin{array}{l} x^2 + y^2 \leq 1 \\ y \geq 0 \end{array} \right\}$$

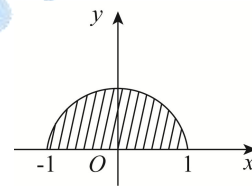
求 $\iint_D x f(x, y) \, d\sigma$



解析:

积分区域 D 如图: $f(x, y) = y\sqrt{1-x^2} + x \iint_D f(x, y) \, dx \, dy$ 两边积分得

$$\iint_D f(x, y) \, dx \, dy = \iint_D y\sqrt{1-x^2} \, dx \, dy + \iint_D f(x, y) \, dx \, dy \cdot \iint_D x \, dx \, dy$$



$$\begin{aligned}
 \iint_D y\sqrt{1-x^2} \, dx \, dy &= 2 \int_0^1 dx \int_0^{\sqrt{1-x^2}} y\sqrt{1-x^2} \, dy \\
 &= 2 \int_0^1 \sqrt{1-x^2} \cdot \frac{1}{2} (1-x^2) \, dx \\
 &= \int_0^1 (1-x^2)^{\frac{3}{2}} \, dx
 \end{aligned}$$

$$\underline{\underline{x = \sin t}} \int_0^{\frac{\pi}{2}} \cos^4 t \, dt = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{3\pi}{16}$$

$$\iint_D x dx dy = 0$$

$$\text{所以 } \iint_D f(x, y) dx dy = \frac{3\pi}{16}$$

$$f(x, y) = y\sqrt{1-x^2} + \frac{3\pi}{16}x$$

$$\text{从而 } \iint_D xf(x, y) dx dy = \iint_D xy\sqrt{1-x^2} dx dy + \iint_D \frac{3\pi}{16}x^2 dx dy$$

$$= \frac{3}{16}\pi \iint_D x^2 dx dy$$

$$= \frac{3}{16}\pi \int_0^1 dx \int_0^{\sqrt{1-x^2}} x^2 dy$$

$$= \frac{3}{16}\pi \int_0^1 x^2 \sqrt{1-x^2} dx$$

$$\underline{\underline{x = \sin t}} \quad \frac{3\pi}{16} \int_0^{\frac{\pi}{2}} \sin^2 t \cos^2 t dt$$

$$= \frac{3\pi}{16} \int_0^{\frac{\pi}{2}} \sin^2 t (1 - \sin^2 t) dt$$

$$= \frac{3\pi}{16} \left(\frac{1}{2} \cdot \frac{\pi}{2} - \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \right)$$

$$= \frac{3\pi}{256}$$

19. $f(x)$ 在 $[0, 2]$ 上具有连续导数, $M = \max\{|f(x)|, x \in [0, 2]\}$

(1) 证 $\exists \xi \in [0, 2] \quad M \leq |f'(\xi)|$

(2) 若 $\forall x \in [0, 2] \quad |f'(x)| \leq M$ 则 $M = 0$

解析:

证明: (1) 由 $M = \max\{|f(x)|, x \in [0, 2]\}$ 知存在 $c \in [0, 2]$, 使 $|f(c)| = M$,

若 $c \in [0, 1]$ 由拉格朗日中值定理得至少存在一点 $\xi \in (0, c)$, 使

$$f'(\xi) = \frac{f(c) - f(0)}{c} = \frac{f(c)}{c}$$

$$\text{从而 } |f'(\xi)| = \frac{|f(c)|}{c} = \frac{M}{c} \geq M$$

若 $c \in (1, 2]$, 同理存在 $\xi \in (c, 2)$ 使

$$f'(\xi) = \frac{f(2) - f(c)}{2 - c} = \frac{-f(c)}{2 - c}$$

$$\text{从而 } |f'(\xi)| = \frac{|f(c)|}{2 - c} = \frac{M}{2 - c} \geq M$$

综上, 存在 $\xi \in (0, 2)$, 使 $|f'(\xi)| \geq M$.

(2) 若 $M > 0$, 则 $c \neq 0, 2$.

由 $f(0) = f(2) = 0$ 及罗尔定理知, 存在 $\eta \in (0, 2)$, 使 $f'(\eta) = 0$,

当 $\eta \in (0, c]$ 时,

$$f(c) - f(0) = \int_0^c f'(x) dx$$

$$M = |f(c)| = |f(c) - f(0)| \leq \int_0^c |f'(x)| dx < Mc,$$

$$\text{又 } f(2) - f(c) = \int_c^2 f'(x) dx$$

$$M = |f(c)| = |f(2) - f(c)| \leq \int_c^2 |f'(x)| dx \leq M(2 - c)$$

于是 $2M < Mc + M(2 - c) = 2M$ 矛盾.

故 $M = 0$.

20. 设二次型 $f(x_1, x_2) = x_1^2 + 4x_1x_2 + 4x_2^2$ 经正交变换 $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Q \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ 化为二次型

$g(y_1, y_2) = ay_1^2 + 4y_1y_2 + by_2^2$, 其中 $a \geq b$.

(1) 求 a, b 的值.

(2) 求正交矩阵 Q .

解析:

$$(1) \text{ 设 } A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} a & 2 \\ 2 & b \end{bmatrix}$$

由题意可知 $Q^T A Q = Q^{-1} A Q = B$.

$\therefore A$ 合同相似于 B

$$\therefore \begin{cases} 1+4 = a+b \\ ab = 4 \end{cases} \quad a \geq b$$

$$\therefore a = 4, \quad b = 1$$

$$(2) |\lambda E - A| = \begin{vmatrix} \lambda - 1 & 2 \\ 2 & \lambda - 4 \end{vmatrix} = \lambda^2 - 5\lambda$$

$\therefore A$ 的特征值为 0, 5

当 $\lambda = 0$ 时, 解 $(0E - A)x = 0$ 得基础解为

$$\alpha_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

当 $\lambda = 5$ 时, 解 $(5E - A)x = 0$ 得基础解为

$$\alpha_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

又 B 的特征值也为 0, 5

当 $\lambda = 0$ 时, 解 $(0E - B)x = 0$ 得

$$\beta_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \alpha_2$$

当 $\lambda = 5$ 时, 解 $(5E - B)x = 0$ 得

$$\beta_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \alpha_1$$

对 α_1, α_2 单位化

$$\gamma_1 = \frac{\alpha_1}{|\alpha_1|} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ 1 \\ \sqrt{5} \end{bmatrix}$$

$$\gamma_2 = \frac{\alpha_2}{|\alpha_2|} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \sqrt{5} \\ -2 \\ \sqrt{5} \end{bmatrix}$$

令 $Q_1 = [\gamma_1, \gamma_2], Q_2 = [\gamma_2, \gamma_1]$

$$\text{则 } Q_1^T A Q_1 = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} = Q_2^T B Q_2$$

故 $Q_2 Q_1^T A Q_1 Q_2^T = B$

可令

$$\begin{aligned} Q &= Q_1 Q_2^T \\ &= \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & -\frac{4}{5} \end{bmatrix} \end{aligned}$$

21. 设 A 为 2 阶矩阵, $P = (\alpha, A\alpha)$, 其中 α 是非零向量且不是 A 的特征向量.

(1) 证明 P 为可逆矩阵

(2) 若 $A^2\alpha + A\alpha - 6\alpha = 0$, 求 $P^{-1}AP$, 并判断 A 是否相似于对角矩阵.

解析:

(1) $\alpha \neq 0$ 且 $A\alpha \neq \lambda\alpha$.

故 α 与 $A\alpha$ 线性无关.

则 $r(\alpha, A\alpha) = 2$

则 P 可逆.

$$AP = A(\alpha, A\alpha)$$

$$= (A\alpha, A^2x)$$

$$= (\alpha A\alpha) \begin{pmatrix} 0 & 6 \\ 1 & -1 \end{pmatrix}$$

$$\text{故 } P^{-1}AP = \begin{pmatrix} 0 & 6 \\ 1 & -1 \end{pmatrix}.$$

(2) 由 $A^2\alpha + A\alpha - 6\alpha = 0$

$$(A^2 + A - 6E)\alpha = 0$$

$$\text{设 } (A+3E)(A-2E)\alpha = 0$$

由 $\alpha \neq 0$ 得 $(A^2 + A - 6E)x = 0$ 有非零解

$$\text{故 } |(A+3E)(A-2E)| = 0$$

$$\text{得 } |A+3E| = 0 \text{ 或 } |A-2E| = 0$$

若 $|A+3E| \neq 0$ 则有 $(A-2E)\alpha = 0$ 故 $A\alpha = 2\alpha$ 与题意矛盾

$$\text{故 } |A+3E| = 0 \text{ 同理可得 } |A-2E| = 0$$

于是 A 的特征值为 $\lambda_1 = -3$ $\lambda_2 = 2$.

A 有 2 个不同特征值故 $A\alpha$ 相似对角化

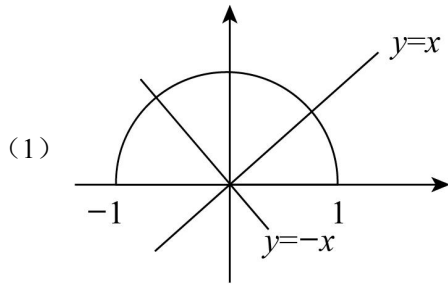
22. 二维随机变量 (X, Y) 在 $D = \{(x, y) | 0 < y < \sqrt{1-x^2}\}$ 上服从均匀分布

$$Z_1 = \begin{cases} 1 & X-Y > 0 \\ 0 & X-Y \leq 0 \end{cases}, \quad Z_2 = \begin{cases} 1 & X+Y > 0 \\ 0 & X+Y \leq 0 \end{cases}$$

(1) 求 (Z_1, Z_2) 联合分布

(2) $\rho_{Z_1 Z_2}$

解析:



$$(x, y) \text{ 服从均匀分布则 } f(x, y) = \begin{cases} \frac{2}{\pi}, & 0 < y < \sqrt{1-x^2} \\ 0, & \text{其他} \end{cases}$$

$$\text{则 } P\{Z_1 = 0, Z_2 = 0\} = P\{X \leq Y, X \leq -Y\} = \frac{1}{4}$$

$$P\{Z_1 = 0, Z_2 = 1\} = P\{X \leq Y, Y > -X\} = \frac{1}{2}$$

$$P\{Z_1 = 1, Z_2 = 0\} = P\{X > Y, X \leq -Y\} = 0$$

$$P\{Z_1 = 1, Z_2 = 1\} = P\{X > Y; X > -Y\} = \frac{1}{4}$$

$$(2) Z_1, Z_2 \text{ 的相关系数 } \rho = \frac{\text{Cov}(Z_1, Z_2)}{\sqrt{DZ_1}\sqrt{DZ_2}}$$

$$\begin{aligned} &= \frac{EZ_1Z_2 - EZ_1EZ_2}{\sqrt{EZ_1^2 - (EZ_1)^2}\sqrt{EZ_2^2 - (EZ_2)^2}} \\ &= \frac{\frac{1}{4} - \frac{1}{4} \cdot \frac{3}{4}}{\sqrt{\frac{1}{4} - \left(\frac{1}{4}\right)^2}\sqrt{\frac{3}{4} - \left(\frac{3}{4}\right)^2}} = \frac{\frac{1}{16}}{\frac{3}{16}} = \frac{1}{3}. \end{aligned}$$

23. 设某种元件的使用寿命 T 的分布函数为

$$F(t) = \begin{cases} 1 - e^{-\left(\frac{t}{\theta}\right)^m}, & t \geq 0, \\ 0, & \text{其他} \end{cases}$$

其中 θ, m 为参数且大于零.

(1) 求概率 $P\{T > t\}$ 与 $P\{T > S+t | T > S\}$, 其中 $S > 0, t > 0$.

(2) 任取 n 个这种元件做寿命试验, 测得它们的寿命分别为 t_1, t_2, \dots, t_n , 若 m 已知, 求

θ 的最大似然估计值 $\hat{\theta}$.

解析:

$$(1) P\{T > t\} = 1 - F(t) = e^{-\left(\frac{t}{\theta}\right)^m}$$

$$P\{T > s+t | T > s\} = P\{T > t\} = e^{-\left(\frac{t}{\theta}\right)^m}$$

$$(2) f(t) = F'(t) = \begin{cases} m \cdot \theta^{-m} \cdot t^{m-1} \cdot e^{-\left(\frac{t}{\theta}\right)^m}, & t \geq 0 \\ 0 & \text{else} \end{cases}$$

$$\text{似然函数 } L(\theta) = \prod_{i=1}^n f(t_i, \theta)$$

$$= \begin{cases} m^n \theta^{-mn} (t_1 \cdots t_n)^{m-1} e^{-\theta^{-m} \sum_{i=1}^n t_i^m} & t_i \geq 0 \\ 0 & \text{else} \end{cases}$$

当 $t_1 \geq 0, t_2 \geq 0, \dots, t_n \geq 0$ 时

$$L(\theta) = m^n \cdot \theta^{-mn} (t_1 \cdots t_n)^{m-1} e^{-\theta^{-m} \sum_{i=1}^n t_i^m}$$

$$\text{取对数 } \ln L(\theta) = n \ln m - mn \ln \theta + (m-1) \sum_{i=1}^n \ln t_i - \theta^{-m} \sum_{i=1}^n t_i^m$$

$$\text{求导数 } \frac{d \ln L(\theta)}{d\theta} = -\frac{mn}{\theta} + m\theta^{-(m+1)} \sum_{i=1}^n t_i^m$$

$$\text{令 } \frac{d \ln L(\theta)}{d\theta} = 0 \text{ 解得 } \theta = \sqrt[m]{\frac{1}{n} \sum_{i=1}^n t_i^m}$$

$$\text{所以 } \theta \text{ 的最大似然估计值 } \hat{\theta} = \sqrt[m]{\frac{1}{n} \sum_{i=1}^n t_i^m}$$